1. *If φ and ψ be differentiable scalar functions of position (x, y, z), then to prove that*

$$\nabla(\varphi \pm \psi) = \nabla\varphi \pm \nabla\psi$$

or

$$\text{grad}\,(\varphi \pm \psi) = \text{grad}\,\varphi \pm \text{grad}\,\psi.$$

Proof. By definition, we have

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}.$$

$\therefore$ grad $(\varphi \pm \psi)$

$$= \nabla(\varphi \pm \psi)$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(\varphi \pm \psi)$$

$$= \vec{i}\frac{\partial}{\partial x}(\varphi \pm \psi) + \vec{j}\frac{\partial}{\partial y}(\varphi \pm \psi) + \vec{k}\frac{\partial}{\partial z}(\varphi \pm \psi)$$

$$= \vec{i}\left(\frac{\partial\varphi}{\partial x} \pm \frac{\partial\psi}{\partial x}\right) + \vec{j}\left(\frac{\partial\varphi}{\partial y} \pm \frac{\partial\psi}{\partial y}\right) + \vec{k}\left(\frac{\partial\varphi}{\partial z} \pm \frac{\partial\psi}{\partial z}\right)$$

$$= \left(\vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}\right) \pm \left(\vec{i}\frac{\partial\psi}{\partial x} + \vec{j}\frac{\partial\psi}{\partial y} + \vec{k}\frac{\partial\psi}{\partial z}\right)$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\varphi \pm \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\psi$$

$$= \nabla\varphi \pm \nabla\psi$$

$$= \text{grad}\,\varphi \pm \text{grad}\,\psi.$$

II. *If $\vec{u}$ and $\vec{v}$ be differentiable vector functions of position (x, y, z), to prove that*

$$\nabla.(\vec{u} \pm \vec{v}) = \nabla.\vec{u} \pm \nabla.\vec{v}$$

or

$$\text{div}\,(\vec{u} \pm \vec{v}) = \text{div}\,\vec{u} \pm \text{div}\,\vec{v}.$$

Proof. By definition, we have

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}.$$

$\therefore$ div $(\vec{u} \pm \vec{v})$

$$= \nabla.(\vec{u} \pm \vec{v})$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).(\vec{u} \pm \vec{v})$$

$$= \vec{i}.\frac{\partial}{\partial x}(\vec{u} \pm \vec{v}) + \vec{j}.\frac{\partial}{\partial y}(\vec{u} \pm \vec{v}) + \vec{k}.\frac{\partial}{\partial z}(\vec{u} \pm \vec{v})$$

$$= \vec{i}.\left(\frac{\partial \vec{u}}{\partial x} \pm \frac{\partial \vec{v}}{\partial x}\right) + \vec{j}.\left(\frac{\partial \vec{u}}{\partial y} \pm \frac{\partial \vec{v}}{\partial y}\right) + \vec{k}.\left(\frac{\partial \vec{u}}{\partial z} \pm \frac{\partial \vec{v}}{\partial z}\right)$$

$$= \left(\vec{i}.\frac{\partial \vec{u}}{\partial x} + \vec{j}.\frac{\partial \vec{u}}{\partial y} + \vec{k}.\frac{\partial \vec{u}}{\partial z}\right) \pm \left(\vec{i}.\frac{\partial \vec{v}}{\partial x} + \vec{j}.\frac{\partial \vec{v}}{\partial y} + \vec{k}.\frac{\partial \vec{v}}{\partial z}\right)$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).\vec{u} \pm \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).\vec{v}$$

$$= \nabla.\vec{u} \pm \nabla.\vec{v}$$

$$= \text{div}\,\vec{u} \pm \text{div}\,\vec{v}.$$

III. *To prove that*

$$\nabla \times (\vec{u} \pm \vec{v}) = \nabla \times \vec{u} \pm \nabla \times \vec{v}$$

or

$$\text{curl}\,(\vec{u} \pm \vec{v}) = \text{curl}\,\vec{u} \pm \text{curl}\,\vec{v}.$$

Proof. By definition, we have

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}.$$

Then curl $(\vec{u} \pm \vec{v})$

$$= \nabla \times (\vec{u} \pm \vec{v})$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial}\right) \times (\vec{u} \pm \vec{v})$$

$$= \vec{i} \times \frac{\partial}{\partial x}(\vec{u} \pm \vec{v}) + \vec{j} \times \frac{\partial}{\partial y}(\vec{u} \pm \vec{v}) + \vec{k} \times \frac{\partial}{\partial z}(\vec{u} \pm \vec{v})$$

$$= \vec{i} \times \left(\frac{\partial \vec{u}}{\partial x} \pm \frac{\partial \vec{v}}{\partial x}\right) + \vec{j} \times \left(\frac{\partial \vec{u}}{\partial y} \pm \frac{\partial \vec{v}}{\partial y}\right) + \vec{k} \times \left(\frac{\partial \vec{u}}{\partial z} \pm \frac{\partial \vec{v}}{\partial z}\right)$$

$$= \left(\vec{i} \times \frac{\partial \vec{u}}{\partial x} + \vec{j} \times \frac{\partial \vec{u}}{\partial y} + \vec{k} \times \frac{\partial \vec{u}}{\partial z}\right) \pm \left(\vec{i} \times \frac{\partial \vec{v}}{\partial x} + \vec{j} \times \frac{\partial \vec{v}}{\partial y} + \vec{k} \times \frac{\partial \vec{v}}{\partial z}\right)$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \times \vec{u} \pm \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \times \vec{v}$$

$$= \nabla \times \vec{u} \pm \nabla \times \vec{v}$$

$$= \text{curl } \vec{u} \pm \text{curl } \vec{v}.$$

IV. *To prove that*

$$\nabla(\varphi\psi) = \varphi\nabla\psi + \psi\nabla\varphi$$

or

$$\textit{grad}\,(\varphi\psi) = \varphi\ \textit{grad}\ \psi + \psi\ \textit{grad}\ \varphi.$$

(M. U. 1987)

Proof. By definition, we have

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}.$$

Then grad $(\varphi\psi) = \nabla(\varphi\psi)$

$$=\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)(\varphi\psi)$$

$$=\vec{i}\frac{\partial}{\partial x}(\varphi\psi)+\vec{j}\frac{\partial}{\partial y}(\varphi\psi)+\vec{k}\frac{\partial}{\partial z}(\varphi\psi)$$

$$=\vec{i}\left(\varphi\frac{\partial\psi}{\partial x}+\psi\frac{\partial\varphi}{\partial x}\right)+\vec{j}\left(\varphi\frac{\partial\psi}{\partial y}+\psi\frac{\partial\varphi}{\partial y}\right)+\vec{k}\left(\varphi\frac{\partial\psi}{\partial z}+\psi\frac{\partial\varphi}{\partial z}\right)$$

$$=\varphi\left(\vec{i}\frac{\partial\psi}{\partial x}+\vec{j}\frac{\partial\psi}{\partial y}+\vec{k}\frac{\partial\psi}{\partial z}\right)+\psi\left(\vec{i}\frac{\partial\varphi}{\partial x}+\vec{j}\frac{\partial\varphi}{\partial y}+\vec{k}\frac{\partial\varphi}{\partial z}\right)$$

$$=\varphi\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\psi+\psi\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\varphi$$

$$=\varphi\nabla\psi+\psi\nabla\varphi$$

$$=\varphi\text{ grad }\psi+\psi\text{ grad }\varphi.$$

V. *To prove that*

$$\nabla\left(\frac{\varphi_1}{\varphi_2}\right)=\frac{\varphi_2\nabla\varphi_1-\varphi_1\nabla\varphi_2}{\varphi_2^2}$$

or

$$grad\left(\frac{\varphi_1}{\varphi_2}\right)=\frac{\varphi_2\, grad\,\varphi_1-\varphi_1\, grad\,\varphi_2}{\varphi_2^2},$$

where φ_1 and φ_2 are two scalar point functions.

Proof. $\text{grad}\left(\frac{\varphi_1}{\varphi_2}\right)=\nabla\left(\frac{\varphi_1}{\varphi_2}\right)$

$$=\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\left(\frac{\varphi_1}{\varphi_2}\right)$$

$$=\vec{i}\frac{\partial}{\partial x}\left(\frac{\varphi_1}{\varphi_2}\right)+\vec{j}\frac{\partial}{\partial y}\left(\frac{\varphi_1}{\varphi_2}\right)+\vec{k}\frac{\partial}{\partial z}\left(\frac{\varphi_1}{\varphi_2}\right)$$

$$=\vec{i}\;\frac{\varphi_2\frac{\partial\varphi_1}{\partial x}-\varphi_1\frac{\partial\varphi_2}{\partial x}}{\varphi_2^2}+\vec{j}\;\frac{\varphi_2\frac{\partial\varphi_1}{\partial y}-\varphi_1\frac{\partial\varphi_2}{\partial y}}{\varphi_2^2}$$

$$+\vec{k}\;\frac{\varphi_2\frac{\partial\varphi_1}{\partial z}-\varphi_1\frac{\partial\varphi_2}{\partial z}}{\varphi_2^2}$$

$$= \frac{1}{\varphi_2^2}\left[\varphi_2\left(\vec{i}\,\frac{\partial \varphi_1}{\partial x} + \vec{j}\,\frac{\partial \varphi_1}{\partial y} + \vec{k}\,\frac{\partial \varphi_1}{\partial z}\right) - \varphi_1\left(\vec{i}\,\frac{\partial \varphi_2}{\partial x} + \vec{j}\,\frac{\partial \varphi_2}{\partial y} + \vec{k}\,\frac{\partial \varphi_2}{\partial z}\right)\right]$$

$$= \frac{\varphi_2 \nabla \varphi_1 - \varphi_1 \nabla \varphi_2}{\varphi_2^2} = \frac{\varphi_2 \text{ grad } \varphi_1 - \varphi_1 \text{ grad } \varphi_2}{\varphi_2^2}.$$

VI. *To prove that*

$$\nabla . (\varphi \vec{u}) = (\nabla \varphi) . \vec{u} + \varphi(\nabla . \vec{u})$$

or

$$\textit{div}\ (\varphi \vec{u}) = (\textit{grad}\ \varphi) . \vec{u} + \varphi\ \textit{div}\ \vec{u}.$$

(P. U. 1985)

Proof. By definition, we have

$$\nabla = \vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}.$$

Then div $(\varphi \vec{u})$

$$= \nabla . (\varphi \vec{u})$$

$$= \left(\vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}\right) . (\varphi \vec{u})$$

$$= \vec{i} . \frac{\partial}{\partial x}(\varphi \vec{u}) + \vec{j} . \frac{\partial}{\partial y}(\varphi \vec{u}) + \vec{k} . \frac{\partial}{\partial z}(\varphi \vec{u})$$

$$= \vec{i} . \left(\frac{\partial \varphi}{\partial x}\vec{u} + \varphi\frac{\partial \vec{u}}{\partial x}\right) + \vec{j} . \left(\frac{\partial \varphi}{\partial y}\vec{u} + \varphi\frac{\partial \vec{u}}{\partial y}\right) + \vec{k} . \left(\frac{\partial \varphi}{\partial z}\vec{u} + \varphi\frac{\partial \vec{u}}{\partial z}\right)$$

$$= \left(\vec{i}\,\frac{\partial \varphi}{\partial x} + \vec{j}\,\frac{\partial \varphi}{\partial y} + \vec{k}\,\frac{\partial \varphi}{\partial z}\right) . \vec{u} + \varphi\left(\vec{i} . \frac{\partial \vec{u}}{\partial x} + \vec{j} . \frac{\partial \vec{u}}{\partial y} + \vec{k} . \frac{\partial \vec{u}}{\partial z}\right)$$

$$= (\nabla \varphi) . \vec{u} + \varphi(\nabla . \vec{u})$$

$$= (\text{grad } \varphi) . \vec{u} + \varphi \text{ div } \vec{u}.$$

VII. *To prove that*

$$\nabla\times(\varphi\vec{u})=(\nabla\varphi)\times\vec{u}+\varphi(\nabla\times\vec{u})$$

or

$$curl\,(\varphi\vec{u})=(grad\,\varphi)\times\vec{u}+\varphi\,curl\,\vec{u}.$$

(M. U. 1986)

Proof. By definition, we have

$$\nabla=\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}.$$

$\therefore$ curl $(\varphi\vec{u})$

$$=\nabla\times(\varphi\vec{u})$$

$$=\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\times(\varphi\vec{u})$$

$$=\vec{i}\times\frac{\partial}{\partial x}(\varphi\vec{u})+\vec{j}\times\frac{\partial}{\partial y}(\varphi\vec{u})+\vec{k}\times\frac{\partial}{\partial z}(\varphi\vec{u})$$

$$=\vec{i}\times\left(\frac{\partial\varphi}{\partial x}\vec{u}+\varphi\frac{\partial\vec{u}}{\partial x}\right)+\vec{j}\times\left(\frac{\partial\varphi}{\partial y}\vec{u}+\varphi\frac{\partial\vec{u}}{\partial y}\right)+\vec{k}\times\left(\frac{\partial\varphi}{\partial z}\vec{u}+\varphi\frac{\partial\vec{u}}{\partial z}\right)$$

$$=\left(\vec{i}\frac{\partial\varphi}{\partial x}+\vec{j}\frac{\partial\varphi}{\partial y}+\vec{k}\frac{\partial\varphi}{\partial z}\right)\times\vec{u}+\varphi\left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\times\vec{u}$$

$$=(\nabla\varphi)\times\vec{u}+\varphi(\nabla\times\vec{u})$$

$$=(\text{grad}\,\varphi)\times\vec{u}+\varphi\,(\text{curl}\,\vec{u}).$$

VIII. *To prove that*

$$\nabla.(\vec{u}\times\vec{v})=\vec{v}.(\nabla\times\vec{u})-\vec{u}.(\nabla\times\vec{v})$$

or

$$div\,(\vec{u}\times\vec{v})=\vec{v}.\,curl\,\vec{u}-\vec{u}.\,curl\,\vec{v}.$$

(Bh. U. 1986; Mith. U. '85)

Proof. By definition, we have

$$\nabla=\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}.$$

$\therefore \quad \text{div}(\vec{u} \times \vec{v})$

$$= \nabla.(\vec{u} \times \vec{v})$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).(\vec{u} \times \vec{v})$$

$$= \vec{i}.\frac{\partial}{\partial x}(\vec{u} \times \vec{v}) + \vec{j}.\frac{\partial}{\partial y}(\vec{u} \times \vec{v}) + \vec{k}.\frac{\partial}{\partial z}(\vec{u} \times \vec{v})$$

$$= \vec{i}.\left(\frac{\partial \vec{u}}{\partial x} \times \vec{v} + \vec{u} \times \frac{\partial \vec{v}}{\partial x}\right) + \vec{j}.\left(\frac{\partial \vec{u}}{\partial y} \times \vec{v} + \vec{u} \times \frac{\partial \vec{v}}{\partial y}\right)$$
$$+ \vec{k}.\left(\frac{\partial \vec{u}}{\partial z} \times \vec{v} + \vec{u} \times \frac{\partial \vec{v}}{\partial z}\right)$$

$$= \left[\vec{i}.\left(\frac{\partial \vec{u}}{\partial x} \times \vec{v}\right) + \vec{j}.\left(\frac{\partial \vec{u}}{\partial y} \times \vec{v}\right) + \vec{k}.\left(\frac{\partial \vec{u}}{\partial z} \times \vec{v}\right)\right]$$
$$+ \left[\vec{i}.\left(\vec{u} \times \frac{\partial \vec{v}}{\partial x}\right) + \vec{j}.\left(\vec{u} \times \frac{\partial \vec{v}}{\partial y}\right) + \vec{k}.\left(\vec{u} \times \frac{\partial \vec{v}}{\partial z}\right)\right]$$

$$= \left[\left(\vec{i} \times \frac{\partial \vec{u}}{\partial x}\right).\vec{v} + \left(\vec{j} \times \frac{\partial \vec{u}}{\partial y}\right).\vec{v} + \left(\vec{k} \times \frac{\partial \vec{u}}{\partial z}\right).\vec{v}\right]$$
$$- \left[\vec{i}.\left(\frac{\partial \vec{v}}{\partial x} \times \vec{u}\right) + \vec{j}.\left(\frac{\partial \vec{v}}{\partial y} \times \vec{u}\right) + \vec{k}.\left(\frac{\partial \vec{v}}{\partial z} \times \vec{u}\right)\right],$$

as in a scalar triple product, the positions of dot and cross can be interchanged and by virtue of

$$\vec{a}.(\vec{b} \times \vec{c}) = -\vec{a}.(\vec{c} \times \vec{b})$$

$$= \left[\vec{i} \times \frac{\partial \vec{u}}{\partial x} + \vec{j} \times \frac{\partial \vec{u}}{\partial y} + \vec{k} \times \frac{\partial \vec{u}}{\partial z}\right].\vec{v}$$
$$- \left[\left(\vec{i} \times \frac{\partial \vec{v}}{\partial x}\right).\vec{u} + \left(\vec{j} \times \frac{\partial \vec{v}}{\partial y}\right).\vec{u} + \left(\vec{k} \times \frac{\partial \vec{v}}{\partial z}\right).\vec{u}\right],$$

as in a scalar triple product, the dot and the cross can be interchanged

$$= \vec{v}.\left\{\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \times \vec{u}\right\}$$
$$- \vec{u}.\left\{\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \times \vec{v}\right\},$$

as the dot product is commutative

$$= \vec{v}.(\nabla \times \vec{u}) - \vec{u}.(\nabla \times \vec{v})$$

$$= \vec{v} \,.\, \text{curl } \vec{u} - \vec{u} \,.\, \text{curl } \vec{v}\,.$$

IX. *To prove that*

$$\nabla \times (\vec{u} \times \vec{v}), \textit{ that is,}$$

$$\textit{curl }(\vec{u} \times \vec{v}) = (\vec{v}.\nabla)\vec{u} - (\vec{u}.\nabla)\vec{v} + \vec{u} \textit{ div } \vec{v} - \vec{v} \textit{ div } \vec{u}\,.$$

(M. U. 1984 H;R. U. 1986; P. U. '86; Mith. U. '86; B. U. '85)

Proof. By definition, we have

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}.$$

$\therefore$ $\text{curl }(\vec{u} \times \vec{v})$

$$= \nabla \times (\vec{u} \times \vec{v})$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \times (\vec{u} \times \vec{v})$$

$$= \vec{i} \times \frac{\partial}{\partial x}(\vec{u} \times \vec{v}) + \vec{j} \times \frac{\partial}{\partial y}(\vec{u} \times \vec{v}) + \vec{k} \times \frac{\partial}{\partial z}(\vec{u} \times \vec{v})$$

$$= \vec{i} \times \left(\vec{u} \times \frac{\partial \vec{v}}{\partial x} + \frac{\partial \vec{u}}{\partial x} \times \vec{v}\right)$$

$$+ \vec{j} \times \left(\vec{u} \times \frac{\partial \vec{v}}{\partial y} + \frac{\partial \vec{u}}{\partial y} \times \vec{v}\right)$$

$$+ \vec{k} \times \left(\vec{u} \times \frac{\partial \vec{v}}{\partial z} + \frac{\partial \vec{u}}{\partial z} \times \vec{v}\right)$$

$$= \left[\vec{i} \times \left(\vec{u} \times \frac{\partial \vec{v}}{\partial x}\right) + \vec{j} \times \left(\vec{u} \times \frac{\partial \vec{v}}{\partial y}\right) + \vec{k} \times \left(\vec{u} \times \frac{\partial \vec{v}}{\partial z}\right)\right]$$

$$+ \left[\vec{i} \times \left(\frac{\partial \vec{u}}{\partial x} \times \vec{v}\right) + \vec{j} \times \left(\frac{\partial \vec{u}}{\partial y} \times \vec{v}\right)\right.$$

$$\left. + \vec{k} \times \left(\frac{\partial \vec{u}}{\partial z} \times \vec{v}\right)\right]$$

$$= \left[\left(\vec{i} \cdot \frac{\partial \vec{v}}{\partial x}\right)\vec{u} - (\vec{i} \cdot \vec{u})\frac{\partial \vec{v}}{\partial x} + \left(\vec{j} \cdot \frac{\partial \vec{v}}{\partial y}\right)\vec{u} - (\vec{j} \cdot \vec{u})\frac{\partial \vec{v}}{\partial y}\right.$$

$$\left. + \left(\vec{k} \cdot \frac{\partial \vec{v}}{\partial z}\right)\vec{u} - (\vec{k} \cdot \vec{u})\frac{\partial \vec{v}}{\partial z}\right]$$

$$+ \left[(\vec{i} \cdot \vec{v})\frac{\partial \vec{u}}{\partial x} - \left(\vec{i} \cdot \frac{\partial \vec{u}}{\partial x}\right)\vec{v} + (\vec{j} \cdot \vec{v})\frac{\partial \vec{u}}{\partial y}\right.$$

$$\left. - \left(\vec{j} \cdot \frac{\partial \vec{u}}{\partial y}\right)\vec{v} + (\vec{k} \cdot \vec{v})\frac{\partial \vec{u}}{\partial z} - \left(\vec{k} \cdot \frac{\partial \vec{u}}{\partial z}\right)\vec{v}\right],$$

using the formula $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

$$= \vec{u}\left\{\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \vec{v}\right\}$$

$$- \left\{\vec{u} \cdot \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\right\}\vec{v}$$

$$+ \left\{\vec{v} \cdot \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\right\}\vec{u}$$

$$- \vec{v}\left\{\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \vec{u}\right\}$$

$$= \vec{u}(\nabla \cdot \vec{v}) - (\vec{u} \cdot \nabla)\vec{v} + (\vec{v} \cdot \nabla)\vec{u} - \vec{v}(\nabla \cdot \vec{u})$$

$$= \vec{u}\,\text{div}\,\vec{v} - (\vec{u} \cdot \nabla)\vec{v} + (\vec{v} \cdot \nabla)\vec{u} - \vec{v}\,\text{div}\,\vec{u}$$

$$= (\vec{v} \cdot \nabla)\vec{u} - (\vec{u} \cdot \nabla)\vec{v} + \vec{u}\,\text{div}\,\vec{v} - \vec{v}\,\text{div}\,\vec{u}.$$

X. To prove that

$$\nabla(\vec{u} \cdot \vec{v})$$
$$= \vec{u} \times (\nabla \times \vec{v}) + \vec{v} \times (\nabla \times \vec{u}) + (\vec{u} \cdot \nabla)\vec{v} + (\vec{v} \cdot \nabla)\vec{u}$$

or

$$\textit{grad}\,(\vec{u} \cdot \vec{v})$$
$$= \vec{u} \times (\textit{curl}\,\vec{v}) + \vec{v} \times (\textit{curl}\,\vec{u}) + (\vec{u} \cdot \nabla)\vec{v} + (\vec{v} \cdot \nabla)\vec{u}.$$

(M. U. 1982 H)

Proof. By definition, we have

$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}.$$

$\therefore$ grad $(\vec{u}\,.\,\vec{v})$

$$= \nabla(\vec{u}\,.\,\vec{v})$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(\vec{u}\,.\,\vec{v})$$

$$= \vec{i}\frac{\partial}{\partial x}(\vec{u}\,.\,\vec{v}) + \vec{j}\frac{\partial}{\partial y}(\vec{u}\,.\,\vec{v}) + \vec{k}\frac{\partial}{\partial z}(\vec{u}\,.\,\vec{v})$$

$$= \vec{i}\left(\vec{u}\,.\,\frac{\partial \vec{v}}{\partial x} + \frac{\partial \vec{u}}{\partial x}\,.\,\vec{v}\right) + \vec{j}\left(\vec{u}\,.\,\frac{\partial \vec{v}}{\partial y} + \frac{\partial \vec{u}}{\partial y}\,.\,\vec{v}\right) + \vec{k}\left(\vec{u}\,.\,\frac{\partial \vec{v}}{\partial z} + \frac{\partial \vec{u}}{\partial z}\,.\,\vec{v}\right)$$

$$= \left\{\left(\vec{u}\,.\,\frac{\partial \vec{v}}{\partial x}\right)\vec{i} + \left(\vec{u}\,.\,\frac{\partial \vec{v}}{\partial y}\right)\vec{j} + \left(\vec{u}\,.\,\frac{\partial \vec{v}}{\partial z}\right)\vec{k}\right\} + \left\{\left(\vec{v}\,.\,\frac{\partial \vec{u}}{\partial x}\right)\vec{i} + \left(\vec{v}\,.\,\frac{\partial \vec{u}}{\partial y}\right)\vec{j} + \left(\vec{v}\,.\,\frac{\partial \vec{u}}{\partial z}\right)\vec{k}\right\} \quad ..\quad (1)$$

But by the well known formula

$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\,.\,\vec{c})\,\vec{b} - (\vec{a}\,.\,\vec{b})\,\vec{c}$$

or $$(\vec{a}\,.\,\vec{b})\,\vec{c} = (\vec{a}\,.\,\vec{c})\,\vec{b} - \vec{a}\times(\vec{b}\times\vec{c})$$

$$= (\vec{a}\,.\,\vec{c})\,\vec{b} + \vec{a}\times(\vec{c}\times\vec{b}),$$

By virtue of this formula, we have

$$\left(\vec{u}\,.\,\frac{\partial \vec{v}}{\partial x}\right)\vec{i} + \left(\vec{u}\,.\,\frac{\partial \vec{v}}{\partial y}\right)\vec{j} + \left(\vec{u}\,.\,\frac{\partial \vec{v}}{\partial z}\right)\vec{k}$$

$$= (\vec{u}\,.\,\vec{i})\frac{\partial \vec{v}}{\partial x} + \vec{u}\times\left(\vec{i}\times\frac{\partial \vec{v}}{\partial x}\right) + (\vec{u}\,.\,\vec{j})\frac{\partial \vec{v}}{\partial y} + \vec{u}\times\left(\vec{j}\times\frac{\partial \vec{v}}{\partial y}\right) + (\vec{u}\,.\,\vec{k})\frac{\partial \vec{v}}{\partial z} + \vec{u}\times\left(\vec{k}\times\frac{\partial \vec{v}}{\partial z}\right)$$